Non-Computability in Graphs

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Computability Theory

- Interested in the nature of computable functions
- Alternatively: recursive functions, lambda calculus, Turing machines, algorithms, etc.
- The meat: how can we talk about non-computable functions?
- Connection to logic: the more non-computable a function is, the more quantifiers we need to define it.

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Computability and Graph Theory

Gain insight into computability theory using graphs as a canvas.
    Example: computable dimension.

Gain insight into graph theory using computability as a tool.
    Example: the Four Color Theorem.

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Computable Dimension

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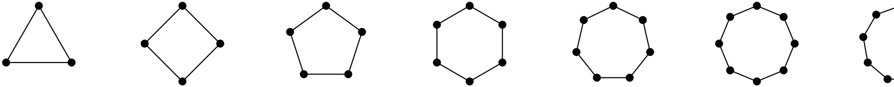
A graph with computable dimension 1:



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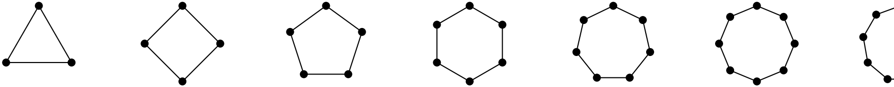

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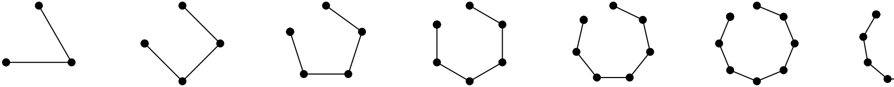

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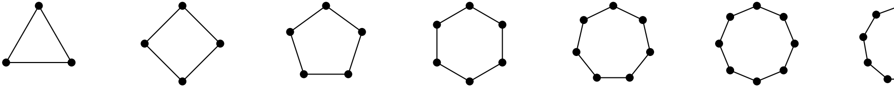

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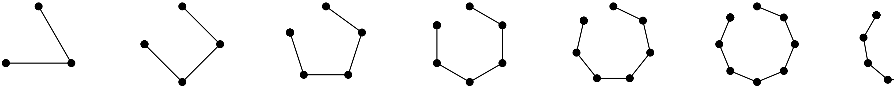

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Computable chromatic number

Any planar graph has a 4-coloring

There are computable planar graphs with no computable k-coloring for any k.

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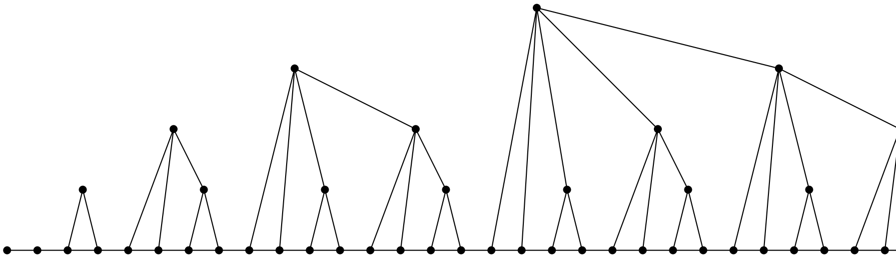
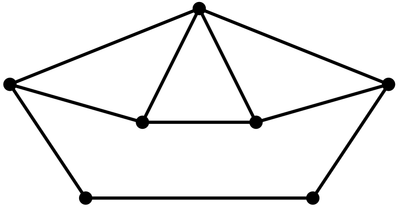


Dominating Sets in Graphs

Given a graph, we look for sets of vertices close to everything.



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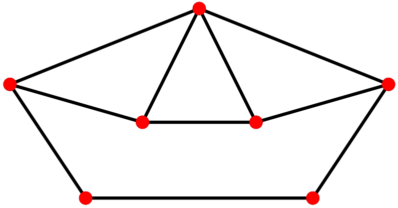

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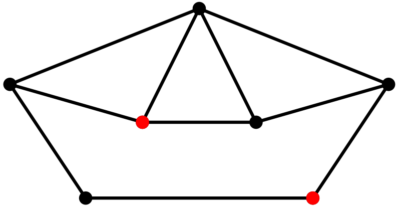

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Domatic Partitions

Definition

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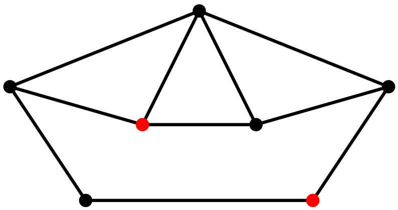

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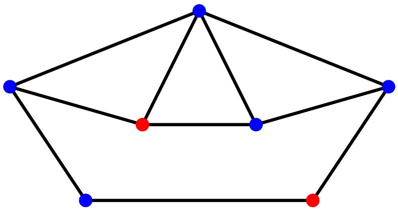

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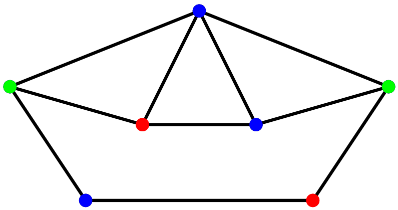

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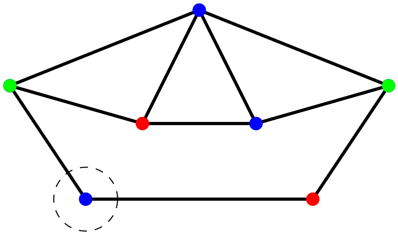

Main Question

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Given a computable graph G with domatic number n, what is the size of the largest computable domatic partition of G?
In other words, what is $d^c(G)$, the computable domatic number?

If $d(G) = 2$ then $d^c(G) = 2$.

Suppose G has a domatic 2-partition (so no isolated vertices).
There is an algorithm which produces a domatic 2-partition.

Vertices: $\{v_0, v_1, v_2, \ldots\}$

Put $v_0 \in A$.

Put $v_n \in B$ iff there is an adjacent vertex $v_k \in A$ (with $k < n$)

A is a dominating set: if $v_n \notin A$ then ...

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There is a computable graph with domatic number 3 but computable domatic number 2.

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Some More Computability Theory

There is an effective list of all (partial) computable functions:

$$\varphi_0, \varphi_1, \varphi_2, \cdots$$

These can be simulated by a universal computable function

We can run these programs "simultaneously" to see if any look like they compute a domatic 3-partition.

Meanwhile, we build a computable graph with a 3-partition

When some φ_e tries to compute a 3-partition, we thwart it.

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The Construction

G will start with copies of K_4, one for each φ_e.

Build G in stages. At each stage, build a new K_4 and check whether φ_e has halted on its copy of K_4.

If φ_e looks like it computes a 3-partition on its K_4, spring the trap!

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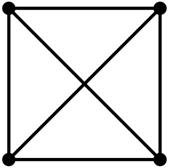

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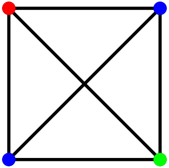

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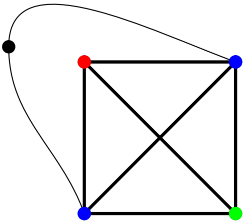

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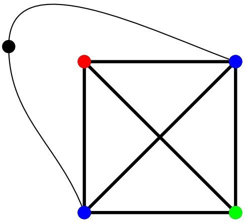

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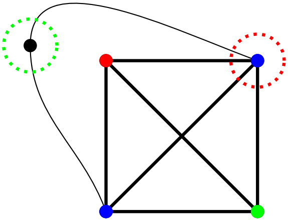

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Use $K_{3(n-2)+1}$ as the trap to diagonalize against all possible computable domatic 3-partitions.

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Why does φ_e partition its trap so soon?

Just because G is computable, doesn't mean we can compute the degree of a given vertex!

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Highly computable graphs

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Does this extra information help φ_e compute a domatic partition?

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Wait for φ_e to partition some fixed vertices. Then act.

Our action cannot change the degree of any vertex in the graph.

φ_e might never partition its vertices, but we don't know that at any finite stage.

We must be able to force φ_e's partition to be wrong, by modifying the graph arbitrarily far away from φ_e's vertices.

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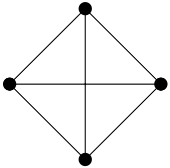
Highly intricate trap

A path:

Every third vertex must be colored the same.

Springing the trap



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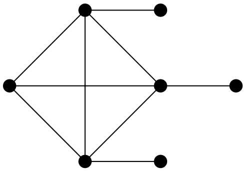

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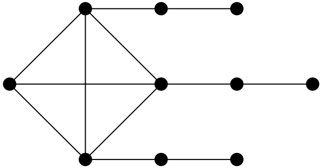

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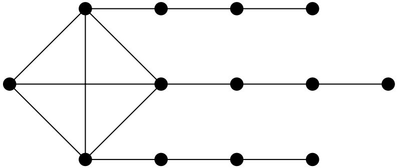

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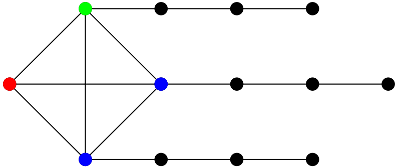

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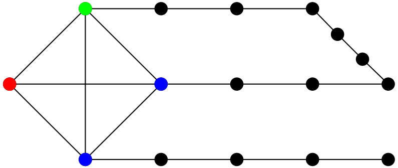

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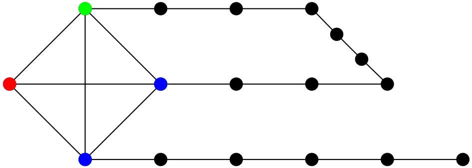

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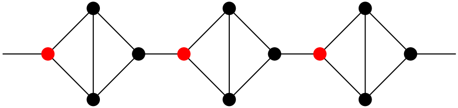

Can we do better?

Is it easier to find smaller domatic partitions in highly computable graphs?

Conjecture

Any highly computable graph with domatic number n has computable domatic number at least $f(n)$.

Maybe $f(n) = n - 1$. Or $f(n) = (n + 1)/2$

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The End

Thanks for listening

Partial results towards and away from the conjecture

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